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XVI.

ON THE GROUP OF AUTOMORPHIC LINEAR TRANSFORMATIONS OF A BILINEAR FORM.

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§ 1. *Alternate Bilinear Form.*

1. In the *Philosophical Transactions* for 1858, Cayley gave the following identity between two matrices, viz.:—

$$(\Omega - \Upsilon) (\Omega + \Upsilon)^{-1} \Omega (\Omega - \Upsilon)^{-1} (\Omega + \Upsilon) = \Omega.$$

From this identity Cayley derives the general solution of the matrical equation $\check{\phi} \Omega \phi = \Omega$, in which Ω is a skew symmetric matrix and $\check{\phi}$ denotes the transverse of ϕ . Thus for ϕ Cayley gives the expression

$$(\Omega - \Upsilon)^{-1} (\Omega + \Upsilon),$$

in which Υ is an arbitrary skew symmetric matrix, but such that

$$|\Omega - \Upsilon| \neq 0.$$

From this expression may be derived

$$\check{\phi} = (\Omega - \Upsilon) (\Omega + \Upsilon)^{-1};$$

therefore, substituting, the above equation is satisfied identically.

As shown by Cayley, the solution of this equation is equivalent to the determination of the automorphic linear transformation of the alternate bilinear form

$$(\Omega) \hat{(x_1, x_2, \dots)} \hat{(y_1, y_2, \dots)},$$

the x 's and y 's being cogredient.

Cayley's expression gives every solution with non-vanishing determinant of the equation $\check{\phi} \Omega \phi = \Omega$, except those of which -1 is a latent root (root of the characteristic equation). If $|\Omega| \neq 0$, and

if ϕ satisfies the above equation, $|\phi| \neq 0$. In what follows it will be assumed that the determinant of the skew symmetric matrix Ω is not zero.

Since the product of any two matrices satisfying the equation $\check{\phi}\Omega\phi = \Omega$ is again a solution of this equation, the matrices satisfying this equation form a group. This group can be generated by the solutions of the equation given by Cayley's expression.* For every solution of the equation $\check{\phi}\Omega\phi = \Omega$ is given by the product of four of Cayley's expressions. See (2) and (10).

The group of solutions of the equation $\check{\phi}\Omega\phi = \Omega$ can be generated by the matrices of the group which differ infinitesimally from the matrix unity; for, see (2) and (5), every solution given by Cayley's expression can be formed by the product into itself an infinite number of times of a solution (given also by Cayley's expression) which differs infinitesimally from the matrix unity.

The group of matrices satisfying the equation $\check{\phi}\Omega\phi = \Omega$ is identical with the group of linear substitutions which automorphically transform the alternate bilinear form

$$(\Omega)(x_1, x_2, \dots)(y_1, y_2, \dots),$$

in which the x 's and y 's are cogredient. Therefore, this group can be generated by the infinitesimal transformations of the group, namely, those differing infinitesimally from the identical substitution.

2. The proof of these theorems relating to the group of solutions of the equation $\check{\phi}\Omega\phi = \Omega$ may be made to depend upon the proof in the special case in which the skew symmetric matrix Ω is also orthogonal (i. e. if $\Omega^2 = -1$).

Thus, let

$$\check{\phi}\Omega\phi = \Omega.$$

If $\Omega^2 \neq -1$, let ω denote any fourth root of $-\Omega^2$ expressible in powers of Ω^2 ; then ω is symmetric and is commutative with Ω . Therefore,

$$\omega^{-1}\check{\phi}\omega\frac{\Omega}{\omega^2}\omega\phi\omega^{-1} = \frac{\Omega}{\omega^2}.$$

If $\psi = \omega\phi\omega^{-1}$, $\Omega' = \frac{\Omega}{\omega^2}$, this becomes

$$\check{\psi}\Omega'\psi = \Omega',$$

* This has been proved otherwise by Frobenius Crelle, 1878, who shows that every solution of the equation $\check{\phi}\Omega\phi = \Omega$ which is not given by Cayley's expression is given by the limit of this expression when the symmetric matrix $\mathbf{T}$ becomes infinite.

in which $\check{\Omega}' = -\Omega'$, and $\Omega'^2 = -1$. If ψ is given by the product of four such expressions as

$$(\Omega' - Y')^{-1} (\Omega' + Y'),$$

in which Y' is symmetric; then since this expression is equal to

$$\begin{aligned} &= \left(\frac{\Omega}{\omega^2} - Y' \right)^{-1} \left(\frac{\Omega}{\omega^2} + Y' \right) \\ &= \omega (\Omega - \omega Y' \omega)^{-1} \omega \cdot \omega^{-1} (\Omega + \omega Y' \omega) \omega^{-1} \\ &= \omega \cdot (\Omega - Y) \cdot \omega^{-1} (\Omega + Y) \cdot \omega^{-1}, \end{aligned}$$

in which $Y = \omega Y' \omega$ is symmetric; therefore $\phi = \omega^{-1} \psi \omega$ is the product of four of Cayley's expressions.

Further, if ϕ is a solution of the equation $\check{\phi} \Omega \phi = \Omega$, given by Cayley's expression, then $\psi = \omega \phi \omega^{-1}$ is a solution of the equation $\check{\psi} \Omega' \psi = \Omega'$ given by Cayley's expression. Therefore, if ψ is equal to the product into itself an infinite number of times of a solution of the equation $\check{\psi} \Omega' \psi = \Omega'$, differing infinitesimally from the matrix unity, ϕ is equal to the product into itself an infinite number of times of a solution of the equation $\check{\phi} \Omega \phi = \Omega$, differing infinitesimally from the matrix unity.*

3. Let now ϕ be any solution of the equation $\check{\phi} \Omega \phi = \Omega$, in which $\check{\Omega} = -\Omega$, and $\Omega^2 = -1$ (and therefore $\check{\Omega} \Omega = 1$). The complete determination of ϕ may be obtained by the consideration of the identity

$$e^{-\Omega \theta} \Omega e^{\theta \Omega} = \Omega,$$

in which $e^{\theta \Omega}$ denotes the exponential series $\sum_0^\infty \frac{(\theta \Omega)^r}{r!}$, convergent for any matrix. Thus, let $-\Omega \theta$ be a polynomial in $\check{\phi}$, satisfying the equation $\check{\phi} = e^{-\Omega \theta}$. We then have $\phi = e^{\theta \Omega}$; but the identity gives $\phi = e^{\theta \Omega}$. Since $-\Omega \theta$ is a polynomial in $\check{\phi}$, $\check{\theta} \Omega$ is a polynomial in ϕ ; and since $\phi^{-1} = \Omega^{-1} \check{\phi} \Omega$, $\theta \Omega = \Omega^{-1} (\Omega \theta) \Omega$ is also a polynomial in ϕ . Consequently, $\check{\theta} \Omega$ and $\theta \Omega$ are commutative. Therefore,

$$e^{(\theta - \check{\theta}) \Omega} = e^{\theta \Omega} e^{-\check{\theta} \Omega} = e^{\theta \Omega} \cdot (e^{\check{\theta} \Omega})^{-1} = \phi \phi^{-1} = 1.$$

* Moreover, if ϕ is a solution of the equation $\check{\phi} \Omega \phi = \Omega$, given by Cayley's expression, it may be given the exponential representation $e^{\theta \Omega}$ in which θ is symmetric. For any symmetric matrix θ , $e^{\theta \Omega}$ is a solution of the equation $\check{\phi} \Omega \phi = \Omega$. Therefore, for any integer n , $e^{\frac{1}{n} \theta \Omega}$ is a solution. But $(e^{\frac{1}{n} \theta})^n = e^\theta = \phi$. Whence the theorem. See note to (5).

4. Conversely, if θ is any matrix for which $\check{\theta}\Omega\theta = \theta\Omega\check{\theta}$ and $e^{(\theta-\check{\theta})\Omega} = 1$, then $\phi = e^{\theta\Omega}$ is a solution of the equation $\check{\phi}\Omega\phi = \Omega$. In particular, if θ is symmetric, $e^{\theta\Omega}$ is a solution of this equation, and so also is $e^{\frac{1}{n}\theta\Omega}$, in which n is any positive integer. But $(e^{\frac{1}{n}\theta\Omega})^n = e^{\theta\Omega}$. Therefore, every expression $e^{\theta\Omega}$ in which θ is symmetric is a solution of the equation $\check{\phi}\Omega\phi = \Omega$, and can be generated by the product into itself an infinite number of times of a matrix which is a solution of this equation, and differs only infinitesimally from the matrix unity.

Let $\phi = e^{\theta\Omega}$ in which θ is symmetric. If the positive integer n is sufficiently great, no odd multiple of $\pi\sqrt{-1}$ is a latent root of $\frac{1}{n}\theta\Omega$; and therefore -1 is not a latent root of $e^{\frac{1}{n}\theta\Omega}$. But $\phi = (e^{\frac{1}{n}\theta\Omega})^n$. Therefore, ϕ is the n th power of a matrix given by Cayley's expression.

5. Every solution of the equation $\check{\phi}\Omega\phi = \Omega$, given by Cayley's expression, can be put in the form $e^{\theta\Omega}$ in which θ is symmetric. Thus, let

$$\begin{aligned}\phi &= (\Omega - Y)^{-1}(\Omega + Y) \\ &= \Omega^{-1} \cdot (1 - Y\Omega^{-1})^{-1}(1 + Y\Omega^{-1}) \cdot \Omega,\end{aligned}$$

in which Y is symmetric. If now $\vartheta' = f(-Y\Omega^{-1})$ is a polynomial in $-Y\Omega^{-1}$, satisfying the equation

$$e^{\vartheta'} = 1 - Y\Omega^{-1},$$

then $\vartheta'' = f(Y\Omega^{-1})$ will satisfy the equation

$$e^{\vartheta''} = 1 + Y\Omega^{-1}.*$$

* Since $|\phi| \neq 0$, and therefore $|1 - Y\Omega^{-1}| \neq 0$, $|1 + Y\Omega^{-1}| \neq 0$, such polynomials in $-Y\Omega^{-1}$ and $Y\Omega^{-1}$ can always be formed; and if $e^{f(-Y\Omega^{-1})} = 1 - Y\Omega^{-1}$, equating the transverse of either member, we have

$$e^{f(\Omega^{-1}Y)} = 1 + \Omega^{-1}Y.$$

That is,

$$\begin{aligned}\Omega^{-1}e^{f(Y\Omega^{-1})}\Omega &= e^{\Omega^{-1}f(Y\Omega^{-1})\Omega} \\ &= e^{f(\Omega^{-1} \cdot Y\Omega^{-1} \cdot \Omega)} \\ &= e^{f(\Omega^{-1}Y)} \\ &= 1 + \Omega^{-1}Y \\ &= \Omega^{-1}(1 + Y\Omega^{-1})\Omega.\end{aligned}$$

Therefore,

$$e^{f(Y\Omega^{-1})} = 1 + Y\Omega^{-1}.$$

Since $\vartheta' \vartheta'' = \vartheta'' \vartheta'$, we have

$$\begin{aligned}\phi &= \Omega^{-1} (e^{\vartheta'})^{-1} e^{\vartheta''} \Omega \\ &= \Omega^{-1} e^{-\vartheta'} e^{\vartheta''} \Omega \\ &= \Omega^{-1} e^{-\vartheta' + \vartheta''} \Omega \\ &= e^{\Omega^{-1}(-\vartheta' + \vartheta'')} \Omega.\end{aligned}$$

But $-\vartheta' + \vartheta''$ is a polynomial in odd powers of $\Upsilon \Omega^{-1}$; and since $\Omega^{-1}(\Upsilon \Omega^{-1})^{2r+1}$ for any positive integer r is symmetric, $\Omega^{-1}(-\vartheta' + \vartheta'') \Omega$ is the product of a symmetric matrix into Ω .

Whence it follows that every solution of the equation $\check{\phi} \Omega \phi = \Omega$ given by Cayley's expression is the product into itself an infinite number of times of a solution which differs only infinitesimally from the matrix unity.*

6. Assuming that $\theta \Omega$ is a polynomial in ϕ satisfying the equation $\check{\phi} \Omega \phi = \Omega$, let $\bar{\phi} = e^{\frac{1}{2}(\theta + \check{\theta})} \Omega$, and $\phi_0 = e^{\frac{1}{2}(\theta - \check{\theta})} \Omega$. Then, by (3),

$$\phi = \bar{\phi} \phi_0 = \phi_0 \bar{\phi};$$

and, by (3) and (4),

$$\check{\phi} \Omega \bar{\phi} = \Omega, \quad \check{\phi}_0 \Omega \phi_0 = \Omega, \quad \phi_0^2 = 1.$$

By (4), $\bar{\phi}$ is the product of matrices given by Cayley's expression. But the matrix θ may be so chosen that no integer multiple of $\pi \sqrt{-1}$ other than zero is a latent root of $\frac{1}{2}(\theta + \check{\theta}) \Omega$, in which case $\bar{\phi}$ is given by Cayley's expression.†

* In the three preceding sections (3), (4), (5), the assumption $\Omega^2 = -1$ has not been employed in the demonstration. The theorems therein given, therefore, hold for any skew symmetric matrix whose determinant does not vanish.

† It is readily shown that $\bar{\phi}$ is given by the square of Cayley's expression. For if some integer multiple of $2\pi \sqrt{-1}$ other than zero is a latent root of $\theta \Omega$, a matrix $\theta' \Omega$ also a polynomial in ϕ can be found, of which no integer multiple of $2\pi \sqrt{-1}$ other than zero is a latent root, such that $\phi = e^{\theta} \Omega = e^{\theta'} \Omega$. But then no integer multiple of $2\pi \sqrt{-1}$ other than zero is a latent root of $\Omega \check{\theta}'$ and therefore of $\check{\theta}' \Omega$. Consequently, no integer multiple of $\frac{\pi \sqrt{-1}}{2}$ other than zero is a latent root either of $\frac{\theta' \Omega}{4}$ or of $\frac{\check{\theta}' \Omega}{4}$. Whence it follows that -1 is not a latent root either of $e^{\frac{1}{4}\theta'} \Omega$ or $e^{\frac{1}{4}\check{\theta}'} \Omega$; and since these matrices are commutative, -1 is not a latent root of their product. Therefore, $e^{\frac{1}{2}(\theta' + \check{\theta}')} \Omega$ is given by Cayley's expression, and $\bar{\phi} = e^{\frac{1}{2}(\theta' + \check{\theta}')} \Omega$ by the square of Cayley's expression.

7. If ϕ is symmetric, $\theta\Omega + \Omega\check{\theta} = 0$, and therefore $\check{\phi}_0 = \phi_0$. Let

$$\Theta = \Omega\phi_0,$$

then $\check{\Theta} = -\Theta$, $\check{\Theta}\Omega\Theta = \Omega$; moreover, $\Theta^2 = -1$. Consequently, Θ is given by Cayley's expression; and therefore

$$\phi = \phi_0\bar{\phi} = \Omega^{-1}\Theta\bar{\phi}$$

is the product of three of Cayley's expressions.

8. If ϕ is orthogonal, it is commutative with Ω ; therefore θ and $\check{\theta}$ are commutative with Ω . Consequently,

$$\begin{aligned}\check{\phi} &= e^{-\frac{1}{2}\Omega(\theta + \check{\theta})} = e^{-\frac{1}{2}(\theta + \check{\theta})\Omega} = \bar{\phi}^{-1} \\ \check{\phi}_0 &= e^{\frac{1}{2}\Omega(\theta - \check{\theta})} = e^{\frac{1}{2}(\theta - \check{\theta})\Omega} = \phi_0.\end{aligned}$$

From the last equation follows

$$\Omega\phi_0 = \phi_0\Omega.$$

Let now

$$\Theta = \Omega\phi_0;$$

then $\check{\Theta} = -\Theta$, $\check{\Theta}\Omega\Theta = \Omega$, and $\Theta^2 = -1$. Consequently, Θ is given by Cayley's expression.

We have $\phi = \Omega^{-1}\Theta\bar{\phi}$; therefore any orthogonal matrix satisfying the equation $\check{\phi}\Omega\phi = \Omega$ is the product of Ω^{-1} into two of Cayley's expressions. But $\Omega^{-1}\phi$ is orthogonal, and is also a solution of this equation, and consequently can be thus expressed. Whence it follows that ϕ is given by the product of two of Cayley's expressions.*

9. This theorem holds if $\Omega^2 \neq -1$. In this case, if $\check{\phi}\Omega\phi = \Omega$ and ϕ is orthogonal, let ω be any fourth root of $-\Omega^2$ expressible in powers of Ω^2 . Then

$$\omega^{-1}\check{\phi}\omega\frac{\Omega}{\omega^2}\omega\phi\omega^{-1} = \Omega.$$

* The two matrices given by Cayley's expression whose product is equal to ϕ can both be taken orthogonal. For in the equation $\phi = \Omega^{-1}\Theta\bar{\phi}$ the matrices Θ and $\bar{\phi}$ are orthogonal. Similarly, if $\Omega^{-1}\phi = \psi = \Omega^{-1}\Theta'\bar{\psi}$, the two matrices Θ' and $\bar{\psi}$ given by Cayley's expression are orthogonal. Therefore, $\phi = \Omega\psi$ is the product of two orthogonal Cayleyan solutions of the equation $\check{\phi}\Omega\phi = \Omega$.

If the expression $(\Omega - \Upsilon)^{-1}(\Omega + \Upsilon)$ is orthogonal $\Omega\Upsilon = \Upsilon\Omega$. Therefore, the orthogonal subgroup of solutions of the equation $\check{\phi}\Omega\phi = \Omega$ in which $\Omega^2 = -1$ is generated by the totality of such expressions for all symmetric matrices Υ commutative with Ω and such that $|\Omega - \Upsilon| \neq 0$.

But ϕ is commutative with Ω , and consequently with ω . Therefore,

$$\check{\phi} \frac{\Omega}{\omega^2} \phi = \frac{\Omega}{\omega^2};$$

and by the last theorem we may put

$$\begin{aligned} \phi &= \left(\frac{\Omega}{\omega^2} - Y'_\alpha \right)^{-1} \left(\frac{\Omega}{\omega^2} + Y'_\alpha \right) \left(\frac{\Omega}{\omega^2} - Y'_\beta \right)^{-1} \left(\frac{\Omega}{\omega^2} + Y'_\beta \right) \\ &= \omega (\Omega - \omega Y'_\alpha \omega)^{-1} (\Omega + \omega Y'_\alpha \omega) (\Omega - \omega Y'_\beta \omega)^{-1} (\Omega + \omega Y'_\beta \omega) \omega^{-1} \\ &= \omega (\Omega - Y_\alpha)^{-1} (\Omega + Y_\alpha) (\Omega - Y_\beta)^{-1} (\Omega + Y_\beta) \omega^{-1}, \end{aligned}$$

in which Y'_α , Y'_β , and therefore $Y_\alpha = \omega Y'_\alpha \omega$, $Y_\beta = \omega Y'_\beta \omega$, are symmetric. Whence it follows that

$$\phi = \omega^{-1} \phi \omega = (\Omega - Y_\alpha)^{-1} (\Omega + Y_\alpha) (\Omega - Y_\beta)^{-1} (\Omega + Y_\beta)$$

is the product of two of Cayley's expressions.*

10. If ϕ is neither symmetric nor orthogonal, let

$$\Theta = \Omega \phi_0;$$

whence it follows that $\check{\Theta} = -\Theta$, and $\check{\Theta} \Omega \Theta = \Omega$.

Let $\theta' \Omega$ be a polynomial in Θ satisfying the equation $\Theta = e^{\theta' \Omega}$; and let

$$\bar{\Theta} = e^{\frac{1}{2}(\theta' + \check{\theta}') \Omega}, \quad \Theta_0 = e^{\frac{1}{2}(\theta' - \check{\theta}') \Omega}.$$

* It is not proved that the two factors of ϕ are orthogonal. But it can be readily proved that ϕ is equal to the product of three matrices given by Cayley's expression, each of which is orthogonal; and therefore the sub-group of orthogonal solutions of the equation $\check{\phi} \Omega \phi = \Omega$ can be generated by the Cayleyan solutions of this sub-group; or, what is the same thing, by the orthogonal matrices commutative with Ω and of which $+1$ is not a latent root.

For, if ϕ is orthogonal, we may put $\phi = \phi_0 \bar{\phi}$, where $\bar{\phi}$ is an orthogonal matrix of which -1 is not a latent root and ϕ_0 is a symmetric orthogonal matrix, both ϕ_0 and $\bar{\phi}$ being polynomials in ϕ . (See these Proceedings, Vol. XXVIII. p. 219.) If now $\check{\phi} \Omega \phi = \Omega$, then $\check{\phi} \Omega \bar{\phi} = \Omega$, and $\phi_0 \Omega = \Omega \phi_0$. From the last equation follows

$$\phi_0 \frac{\Omega}{\omega^2} = \frac{\Omega}{\omega^2} \phi_0.$$

Therefore, by (8)

$$\begin{aligned} \phi_0 &= \left(\frac{\Omega}{\omega^2} \right)^{-1} \left(\frac{\Omega}{\omega^2} - \Upsilon \right)^{-1} \left(\frac{\Omega}{\omega^2} + \Upsilon \right) \\ &= \omega (\Omega - \omega^2)^{-1} (\Omega + \omega^2) (\Omega - \omega \Upsilon \omega)^{-1} (\Omega + \omega \Upsilon \omega) \omega^{-1}. \end{aligned}$$

Therefore, $\phi_0 = \omega^{-1} \phi_0 \omega$ is the product of two of Cayley's expressions. Moreover, the first factor, namely, $(\Omega - \omega^2)^{-1} (\Omega + \omega^2)$ is orthogonal; and, since ϕ_0 is orthogonal, the second factor of ϕ_0 is also orthogonal. Therefore, $\phi = \phi_0 \bar{\phi}$ is the product of three solutions of the equation $\check{\phi} \Omega \phi = \Omega$ given by Cayley's expression, each of which is orthogonal.

Then

$$\check{\Theta} \Omega \bar{\Theta} = \Omega, \check{\Theta}_0 \Omega \Theta_0 = \Omega, \Theta_0^2 = 1;$$

and, if θ' is properly chosen, -1 is not a latent root of $\bar{\Theta}$, which is then given by Cayley's expression.

Since Θ is skew symmetric, $\theta' \Omega$, $\check{\theta}' \Omega$, $\Omega \theta'$, $\Omega \check{\theta}'$, are commutative. Therefore,

$$\check{\Theta}_0 \Theta_0 = \Theta_0 \check{\Theta}_0.$$

Whence follows

$$(\check{\Theta}_0 \Theta_0)^2 = \check{\Theta}_0^2 \Theta_0^2 = 1.$$

Let now

$$\Omega \Theta_0 = \Gamma;$$

then $\check{\Gamma} = -\Gamma$, and $\check{\Gamma} \Omega \Gamma = \Omega$. We also have

$$\check{\Theta}_0 \Theta_0 = -\Gamma^2;$$

and therefore

$$\Gamma^4 = 1.$$

A symmetric matrix ϖ of non-zero determinant can be found to satisfy the equation $\check{\Gamma} \varpi \Gamma = \varpi$. And if $\varpi^{\frac{1}{2}}$ denotes any symmetric square root of ϖ , the last equation may be written

$$(\varpi^{\frac{1}{2}})^{-1} \check{\Gamma} \varpi^{\frac{1}{2}} \cdot \varpi^{\frac{1}{2}} \Gamma (\varpi^{\frac{1}{2}})^{-1} = 1;$$

which, if $\psi = \varpi^{\frac{1}{2}} \Gamma (\varpi^{\frac{1}{2}})^{-1}$, becomes

$$\check{\psi} \psi = 1.$$

Therefore,

$$\Gamma = (\varpi^{\frac{1}{2}})^{-1} \psi \varpi^{\frac{1}{2}},$$

in which ψ is orthogonal. But, since $\check{\Gamma} \Omega \Gamma = \Omega$, ψ is commutative with $(\varpi^{\frac{1}{2}})^{-1} \Omega (\varpi^{\frac{1}{2}})^{-1}$. Therefore, by (9) ψ is equal to the product of two expressions of the form

$$\begin{aligned} & [(\varpi^{\frac{1}{2}})^{-1} \Omega (\varpi^{\frac{1}{2}})^{-1} - Y']^{-1} [(\varpi^{\frac{1}{2}})^{-1} \Omega (\varpi^{\frac{1}{2}})^{-1} + Y'] \\ &= \varpi^{\frac{1}{2}} (\Omega - \varpi^{\frac{1}{2}} Y' \varpi^{\frac{1}{2}})^{-1} \varpi^{\frac{1}{2}} \cdot (\varpi^{\frac{1}{2}})^{-1} (\Omega + \varpi^{\frac{1}{2}} Y' \varpi^{\frac{1}{2}}) (\varpi^{\frac{1}{2}})^{-1} \\ &= \varpi^{\frac{1}{2}} (\Omega - Y)^{-1} (\Omega + Y) (\varpi^{\frac{1}{2}})^{-1}, \end{aligned}$$

in which Y' and therefore $Y = \varpi^{\frac{1}{2}} Y' \varpi^{\frac{1}{2}}$, is symmetric. Whence it follows that $\Gamma = (\varpi^{\frac{1}{2}})^{-1} \psi \varpi^{\frac{1}{2}}$ is the product of two of Cayley's expressions. Therefore,

$$\begin{aligned} \phi &= \phi_0 \bar{\phi} = \Omega^{-1} \Theta \bar{\phi} = \Omega^{-1} \Theta_0 \bar{\Theta} \bar{\phi} \\ &= -\Gamma \bar{\Theta} \bar{\phi} \end{aligned}$$

is equal to the negative of the product of four of Cayley's expressions. Since $-\phi$ is also a solution of the equation $\check{\phi} \Omega \phi = \Omega$, it is also expressible thus. Whence it follows that ϕ is given by the product of four of Cayley's expressions.

11. If ϕ is real, θ may be so chosen that $\bar{\phi}$ and ϕ_0 are real; and then, if Ω is real, $\Theta = \Omega^{-1}\phi_0$ is real. But then, since Θ is real and skew symmetric, -1 is not a latent root of Θ , which is therefore given by Cayley's expression.

Whence it follows that any real matrix ϕ satisfying the equation $\check{\phi}\Omega\phi = \Omega$, in which Ω is real, is the product of Ω^{-1} into two of Cayley's expressions. But $\Omega^{-1}\phi$ is then also a real solution of this equation, and can therefore be thus expressed. Consequently, ϕ is equal to the product of two of Cayley's expressions.*

In each of the factors of ϕ given by Cayley's expression, the symmetric matrix may be taken real.

12. These theorems may be extended to the real solutions of the equation $\check{\phi}\Omega\phi = \Omega$, in which Ω is any real skew symmetric matrix whose determinant does not vanish. For if Ω is real, the latent roots of $-\Omega^2$ are all positive. Therefore, if ω denotes a fourth root of $-\Omega^2$ expressible in powers of Ω^2 , ω may be taken real. If now ϕ is a real solution of the above equation, we have

$$\omega^{-1}\check{\phi}\omega\frac{\Omega}{\omega^2}\omega\phi\omega^{-1} = \frac{\Omega}{\omega^2},$$

in which $\omega\phi\omega^{-1}$ and $\frac{\Omega}{\omega^2}$ are real. Therefore, by the preceding theorem,

$$\begin{aligned}\omega\phi\omega^{-1} &= \left(\frac{\Omega}{\omega^2} - Y'_\alpha\right)^{-1} \left(\frac{\Omega}{\omega^2} + Y'_\alpha\right) \left(\frac{\Omega}{\omega^2} - Y'_\beta\right)^{-1} \left(\frac{\Omega}{\omega^2} + Y'_\beta\right) \\ &= \omega(\Omega - Y_\alpha)^{-1}(\Omega + Y_\alpha)(\Omega - Y_\beta)^{-1}(\Omega + Y_\beta)\omega^{-1},\end{aligned}$$

in which Y'_α, Y'_β , and therefore $Y_\alpha = \omega Y'_\alpha \omega$, $Y_\beta = \omega Y'_\beta \omega$, are real symmetric matrices.

Thus it appears that the group of real solutions of the equation $\check{\phi}\Omega\phi = \Omega$ can be generated by the Cayleyan solutions of this group, that is, by the totality of expressions of the form $(\Omega - Y)^{-1}(\Omega + Y)$, in which Y is a real symmetric matrix such that $|\Omega - Y| \neq 0$.

§ 2. Bilinear Form, neither Symmetric nor Alternate.

1. The problem to determine the most general solution of the equation $\check{\phi}\Omega\phi = \Omega$, in which Ω is neither symmetric nor skew symmetric, may, provided neither ± 1 is a latent root of $\Omega\check{\Omega}^{-1}$, be reduced to the solution of two algebraic equations of the n th degree.

* See ante, p. 178.

Thus, let

$$2 \Omega_0 = \Omega + \check{\Omega}, \quad 2 \Omega_1 = \Omega - \check{\Omega}.$$

Then, if ϕ satisfies the above equation, we have, as shown by Cayley,

$$\check{\phi} \Omega_0 \phi = \Omega_0, \quad \check{\phi} \Omega_1 \phi = \Omega_1;$$

and conversely.

If neither ± 1 is a latent root of $\Omega \check{\Omega}^{-1}$, then $|\Omega_0| \neq 0$, $|\Omega_1| \neq 0$. Since $|\Omega_0| \neq 0$, Ω_0 has a square root expressible by Sylvester's formula in powers of Ω_0 . For the determination of this square root the solution of the algebraic equation $|\Omega_0 - g| = 0$ is requisite. If $\Omega_0^{\frac{1}{2}}$ denotes any symmetric square root of Ω_0 , the equation $\check{\phi} \Omega_0 \phi = \Omega_0$ may be written

$$(\Omega_0^{\frac{1}{2}})^{-1} \check{\phi} \Omega_0^{\frac{1}{2}} \cdot \Omega_0^{\frac{1}{2}} \phi (\Omega_0^{\frac{1}{2}})^{-1} = 1,$$

which, if $\psi = \Omega_0^{\frac{1}{2}} \phi (\Omega_0^{\frac{1}{2}})^{-1}$, becomes

$$\check{\psi} \psi = 1.$$

Therefore, the most general expression for the matrix ϕ satisfying the above equation is

$$\Omega_0^{-\frac{1}{2}} \psi \Omega_0^{\frac{1}{2}},$$

in which ψ is an arbitrary orthogonal matrix.

If $\check{\phi} \Omega \phi = \Omega$, we must also have $\check{\phi} \Omega_1 \phi = \Omega_1$. Therefore,

$$\Omega_0^{\frac{1}{2}} \check{\psi} \Omega_0^{-\frac{1}{2}} \Omega_1 \Omega_0^{-\frac{1}{2}} \psi \Omega_0^{\frac{1}{2}} = \Omega_1,$$

$$\check{\psi} \Omega_0^{-\frac{1}{2}} \Omega_1 \Omega_0^{-\frac{1}{2}} \psi = \Omega_0^{-\frac{1}{2}} \Omega_1 \Omega_0^{-\frac{1}{2}}.$$

But (9) § 1 ψ is given by the product of the three matrices.

$$\begin{aligned} (\Omega_0^{-\frac{1}{2}} \Omega_1 \Omega_0^{-\frac{1}{2}} - Y'_\alpha)^{-1} (\Omega_0^{-\frac{1}{2}} \Omega_1 \Omega_0^{-\frac{1}{2}} + Y'_\alpha) \\ = \Omega_0^{\frac{1}{2}} (\Omega_1 - \Omega_0^{\frac{1}{2}} Y'_\alpha \Omega_0^{\frac{1}{2}})^{-1} (\Omega_1 + \Omega_0^{\frac{1}{2}} Y'_\alpha \Omega_0^{\frac{1}{2}}) \Omega_0^{-\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} (\Omega_0^{-\frac{1}{2}} \Omega_1 \Omega_0^{-\frac{1}{2}} - Y'_\beta)^{-1} (\Omega_0^{-\frac{1}{2}} \Omega_1 \Omega_0^{-\frac{1}{2}} + Y'_\beta) \\ = \Omega_0^{\frac{1}{2}} (\Omega_1 - \Omega_0^{\frac{1}{2}} Y'_\beta \Omega_0^{\frac{1}{2}})^{-1} (\Omega_1 + \Omega_0^{\frac{1}{2}} Y'_\beta \Omega_0^{\frac{1}{2}}) \Omega_0^{-\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} (\Omega_0^{-\frac{1}{2}} \Omega_1 \Omega_0^{-\frac{1}{2}} - Y'_\gamma)^{-1} (\Omega_0^{-\frac{1}{2}} \Omega_1 \Omega_0^{-\frac{1}{2}} + Y'_\gamma) \\ = \Omega_0^{\frac{1}{2}} (\Omega_1 - \Omega_0^{\frac{1}{2}} Y'_\gamma \Omega_0^{\frac{1}{2}})^{-1} (\Omega_1 + \Omega_0^{\frac{1}{2}} Y'_\gamma \Omega_0^{\frac{1}{2}}) \Omega_0^{-\frac{1}{2}}, \end{aligned}$$

in which Y'_α , Y'_β , Y'_γ , are commutative with $\Omega_0^{-\frac{1}{2}} \Omega_1 \Omega_0^{-\frac{1}{2}}$.

If we put $Y_\alpha = \Omega_0^{\frac{1}{2}} Y'_\alpha \Omega_0^{\frac{1}{2}}$, $Y_\beta = \Omega_0^{\frac{1}{2}} Y'_\beta \Omega_0^{\frac{1}{2}}$, $Y_\gamma = \Omega_0^{\frac{1}{2}} Y'_\gamma \Omega_0^{\frac{1}{2}}$, we have for $\phi = \Omega_0^{-\frac{1}{2}} \psi \Omega_0^{\frac{1}{2}}$ the expression

$$(\Omega_1 - Y_\alpha)^{-1} (\Omega_1 + Y_\alpha) \cdot (\Omega_1 - Y_\beta)^{-1} (\Omega_1 + Y_\beta) (\Omega_1 - Y_\gamma)^{-1} (\Omega_1 + Y_\gamma).$$

Therefore, the group of solutions of the equation $\check{\phi} \Omega \phi = \Omega$ is a sub-group of the equation $\check{\phi} \Omega_1 \phi = \Omega_1$, generated by the totality of expressions

$$(\Omega_1 - Y)^{-1} (\Omega_1 + Y),$$

in which the symmetric matrix Y is subject to the condition that $\Omega_0^{-\frac{1}{2}} Y \Omega_0^{-\frac{1}{2}}$ is commutative with $\Omega_0^{-\frac{1}{2}} \Omega_1 \Omega_0^{-\frac{1}{2}}$, and is such that the determinant of $|\Omega - Y| \neq 0$.

To find the most general skew symmetric matrix $\Omega_0^{-\frac{1}{2}} Y \Omega_0^{-\frac{1}{2}}$ commutative with $\Omega_0^{-\frac{1}{2}} \Omega_1 \Omega_0^{-\frac{1}{2}}$ requires in general the solution of an algebraic equation of the n th degree.*

Again, since ψ is the product of three orthogonal matrices of which -1 is not a latent root and which are commutative with Ω_1 , the group of solutions of the equation $\phi \Omega \phi = \Omega$ is a sub-group of the group of solutions of the equation $\phi \Omega_0 \phi = \Omega$ generated by the totality of expressions

$$\Omega_0^{-\frac{1}{2}} (1 - Y)^{-1} (1 + Y) \Omega_0^{\frac{1}{2}} = (\Omega_0 - \Omega_0^{\frac{1}{2}} Y \Omega_0^{\frac{1}{2}})^{-1} (\Omega_0 + \Omega_0^{\frac{1}{2}} Y \Omega_0^{\frac{1}{2}}),$$

in which Y is a skew symmetric matrix commutative with Ω_1 and such that the determinant of $Y - 1$ is not zero.

The sub-group consisting of the orthogonal solutions of the equation $\phi \Omega \phi = \Omega$ is the group of orthogonal matrices commutative with Ω_0 and Ω_1 . It has been shown that the group of orthogonal matrices commutative with Ω_1 is generated by the totality of expressions $(1 - Y)^{-1} (1 + Y)$ in which Y is a skew symmetric matrix commutative with Ω_1 . This group contains only proper orthogonal matrices; and it is readily proved that the group of orthogonal matrices commutative with Ω_0 is generated by the totality of expressions $(1 - Y)^{-1} (1 + Y)$ in which Y is commutative with Ω_0 and such that $|Y - 1| \neq 0$. Therefore the group of orthogonal solutions of the equation $\phi \Omega \phi = \Omega$ can be generated by the totality of orthogonal matrices of which -1 is not a latent root that are commutative with Ω ; that is, by the totality of expressions $(1 - Y)^{-1} (1 + Y)$, in which Y is a skew symmetric matrix commutative with Ω and such that $|Y - 1| \neq 0$.

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* If the fundamental syzygy in $\Omega_0^{-\frac{1}{2}} \Omega_1 \Omega_0^{-\frac{1}{2}}$ is of order equal to the order of this matrix, the most general matrix commutative with $\Omega_0^{-\frac{1}{2}} \Omega_1 \Omega_0^{-\frac{1}{2}}$ is a rational integral function of this matrix. This is the case if the roots of the equation $|\Omega - g \Omega| = 0$ are all distinct. The fundamental syzygy in any matrix ϕ is the polynomial in powers of ϕ of lowest order that vanishes.